

# A NOTE ON THE $\gamma$ -COEFFICIENTS OF THE “TREE EULERIAN POLYNOMIAL”

RAFAEL S. GONZÁLEZ D’LEÓN\*

**ABSTRACT.** We consider the generating polynomial of the number of rooted trees on the set  $\{1, 2, \dots, n\}$  counted by the number of descending edges (a parent with a greater label than a child). This polynomial is an extension of the descent generating polynomial of the set of permutations of a totally ordered  $n$ -set, known as the Eulerian polynomial. We show how this extension shares some of the properties of the classical one. B. Drake proved that this polynomial factors completely over the integers. From his product formula it can be concluded that this polynomial has positive coefficients in the  $\gamma$ -basis and we show that a formula for these coefficients can also be derived. We discuss various combinatorial interpretations of these positive coefficients in terms of leaf-labeled binary trees and in terms of the Stirling permutations introduced by Gessel and Stanley. These interpretations are derived from previous results of the author and Wachs related to the poset of weighted partitions and the free multibracketed Lie algebra.

## 1. INTRODUCTION

A *labeled rooted tree*  $T$  on the set  $[n] := \{1, 2, \dots, n\}$  is a tree whose nodes or vertices are the elements of  $[n]$  and such that one of its nodes has been distinguished and called the *root*. For nodes  $x$  and  $y$  in  $T$  we say that  $x$  is the *child* of  $y$  or  $y$  is the *parent* of  $x$  if  $y$  is the first node following  $x$  in the unique path from  $x$  to the root of  $T$  and we say that  $y = p(x)$ . Nodes that have children are said to be *internal* otherwise we call a node without children a *leaf*. If  $y$  is the parent of  $x$ , we say that the edge  $\{x, y\}$  of  $T$  is *descending* (and we call  $x$  a *descent* of  $T$ ) if the label of  $y$  is greater than the label of  $x$ . We denote  $\text{des}(T)$  the number of descents in  $T$ . Figure 1 shows all the rooted trees on  $[3]$  grouped by the number of descents. We draw the trees with the convention that parents come higher than their children and the root is the highest node. We denote  $\mathcal{T}_n$  the set of rooted trees on  $[n]$  and  $\mathcal{T}_{n,i}$  the set of trees in  $\mathcal{T}_n$  with exactly  $i$  descents.

For a given  $n \geq 1$  define

---

\*This work was supported by NSF Grant DMS 1202755.

$$(1.1) \quad T_n(t) := \sum_{T \in \mathcal{T}_n} t^{\text{des}(T)} = \sum_{i=0}^{n-1} |\mathcal{T}_{n,i}| t^i,$$

the descent generating polynomial of  $\mathcal{T}_n$ . We call  $T_n(t)$  the *tree Eulerian polynomial* in analogy with the classical polynomial  $A_n(t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{des}(\sigma)}$ , that is the descent generating polynomial of the set  $\mathfrak{S}_n$  of permutations of  $[n]$ . We can identify the permutations in  $\mathfrak{S}_n$  with the set of rooted trees on  $[n]$  that have  $n - 1$  internal nodes, each of them having a unique child (and so containing a unique leaf). It is not hard to see that our definition of a descent on this set of trees coincides with the classical definition of descent in a permutation so the polynomial  $T_n(t)$  is an extension of the polynomial  $A_n(t)$ . The polynomials  $A_n(t)$  have been extensively studied in the literature and are known with the name of *Eulerian polynomials* since Euler was one of the first in studying them (see [15]). The Eulerian polynomial  $A_n(t) = \sum_{k=0}^{n-1} A_{n,i} t^i$  have degree  $n - 1$  and its coefficients satisfy the relation

$$(1.2) \quad A_{n,i} = A_{n,n-1-i}.$$

For example, the Eulerian polynomial for  $n = 3$  is  $A_3(t) = 1 + 4t + t^2$ . A polynomial that satisfies Equation 1.2 is called *symmetric* or *palindromic*. It is a simple observation that a symmetric polynomial  $f(t) = \sum_{k=0}^d f_i t^i$  of degree  $d$  with  $f_i \in \mathbb{Z}$  can be written in the form

$$(1.3) \quad \sum_{i=0}^d f_i t^i = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_i t^i (1+t)^{d-2i},$$

where the coefficients  $\gamma_i \in \mathbb{Z}$ , i.e., the set  $\{t^i(1+t)^{d-2i}\}_{i=0}^{\lfloor \frac{d}{2} \rfloor}$ , where  $\lfloor \cdot \rfloor$  is the integer floor function, is a basis (known as the  $\gamma$ -basis) for the space of symmetric polynomials of degree  $d$  with integer coefficients. If  $\gamma_i \geq 0$  then we say that the polynomial  $f(t)$  is  $\gamma$ -positive. It is known that  $A_n(t)$  is  $\gamma$ -positive and that its coefficients  $\gamma_i$  have a nice combinatorial interpretation. Indeed, let  $\hat{\mathfrak{S}}_n$  be the set of permutations in  $\mathfrak{S}_n$  that have no two adjacent descents and no descent in the last position. In [13], Shapiro, Woan and Getu show that

$$\gamma_j = \{|\sigma \in \hat{\mathfrak{S}}_n \mid \text{des}(\sigma) = j|\}.$$

For example,  $A_3(t) = (1+t)^2 + 2t$  with  $\gamma_0 = 1$  and  $\gamma_1 = 2$ . The permutations in  $\hat{\mathfrak{S}}_3$  are, 123 with no descents and; 213 and 312 with one descent. Gal [6] and Brändén [1, 2] have introduced the use of the

$\gamma$ -basis in different contexts. Gal conjectured that the  $\gamma$ -coefficients of the  $h$ -polynomial of a flag simple polytope are all nonnegative. In particular,  $A_n(t)$  is the  $h$ -vector of the permutahedron that is a flag simple polytope so Gal's conjecture is confirmed in this case. Postnikov, Reiner and Williams [12] have confirmed Gal's conjecture for the family of chordal nestohedra that is a large family of flag simple polytopes. For more information about  $\gamma$ -positivity see [3].

We will show that the properties discussed above for the Eulerian polynomial  $A_n(t)$  are also shared by the polynomial  $T_n(t)$  in a similar fashion. The degree of  $T_n(t)$  is also  $n - 1$  and it is easy to see from the definition of a descent that

$$(1.4) \quad |\mathcal{T}_{n,i}| = |\mathcal{T}_{n,n-1-i}|,$$

so  $T_n(t)$  is also symmetric. Indeed there is a natural bijection  $\mathcal{T}_{n,i} \simeq \mathcal{T}_{n,n-1-i}$  where the image of a labeled rooted tree  $T \in \mathcal{T}_{n,i}$ , is the tree in  $\mathcal{T}_{n,n-1-i}$  with the same shape of  $T$  but where each label  $i$  has been replaced by  $n + 1 - i$ . For the example in Figure 1,  $T_3(t) = 2 + 5t + 2t^2$ .

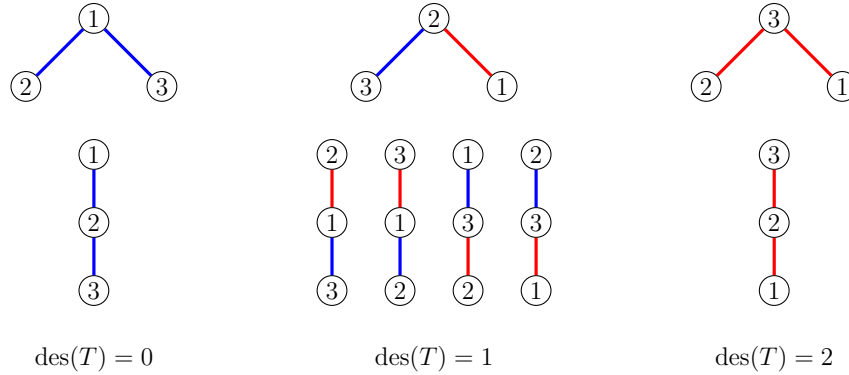


FIGURE 1. All labeled rooted trees on [3]

In [5] Drake proves the following nice product formula for  $T_n(t)$ .

**Theorem 1.1** ([5, Example 1.7.2]). *For  $n \geq 1$ ,*

$$(1.5) \quad \sum_{i=0}^{n-1} |\mathcal{T}_{n,i}| t^i = \prod_{i=1}^{n-1} ((n - i) + it).$$

In particular, setting  $t = 1$  in (1.5) reduces to the classical formula  $|\mathcal{T}_n| = n^{n-1}$ . Equation (1.5) implies that all the roots of this polynomial are real and negative. It is known and not difficult to show that a real-rooted symmetric polynomial with positive real coefficients is  $\gamma$ -positive (see [3, 6]). For example  $T_3(t) = 2(1 + t)^2 + t$ , so  $\gamma_0 = 2$  and  $\gamma_1 = 1$ . Although  $T_n(t)$  is not in general an  $h$ -vector of a convex

polytope (for example  $h_0 \neq 1$  for  $n \geq 3$ ), it is of interest to find combinatorial formulas and interpretations of positive  $\gamma$ -coefficients of general symmetric polynomials. Let  $\gamma_j(T_n(t))$  denote the  $j$ -th gamma coefficient of the symmetric polynomial  $T_n(t)$ .

Equation (1.5) can be used to find a formula for the coefficients  $\gamma_j$  of  $T_n(t)$ .

**Theorem 1.2.** *For  $n \geq 1$ ,*

$$(1.6) \quad \gamma_j(T_n(t)) = \begin{cases} \sum_{\substack{J \subset [\frac{n-1}{2}]} i \in J} \prod (n-2i)^2 \prod_{s \in [\frac{n-1}{2}] \setminus J} s(n-s) & \text{if } n \text{ is odd} \\ \frac{n}{2} \sum_{\substack{J \subset [\frac{n-2}{2}] \\ |J|=j}} \prod_{i \in J} (n-2i)^2 \prod_{s \in [\frac{n-2}{2}] \setminus J} s(n-s) & \text{if } n \text{ is even} \end{cases}$$

*Proof.* If we multiply

$$\begin{aligned} (n-i+it)(i+(n-i)t) &= (n-i)i + [(n-i)^2 + i^2]t + (n-i)it^2 \\ &= (n-i)i(1+t^2) + [(n-i)^2 + i^2]t \\ &= (n-i)i(1+2t+t^2) + [(n-i)^2 - 2i(n-i) + i^2]t \\ &= (n-i)i(1+t)^2 + (n-2i)^2t. \end{aligned}$$

Equation 1.5 can be written as

$$\mathcal{T}(t) = \begin{cases} \prod_{i=1}^{\frac{n-1}{2}} [(n-i)i(1+t)^2 + (n-2i)^2t] & \text{if } n \text{ is odd,} \\ \frac{n}{2}(1+t) \prod_{i=1}^{\frac{n-2}{2}} [(n-i)i(1+t)^2 + (n-2i)^2t] & \text{if } n \text{ is even,} \end{cases}$$

implying Formula 1.6.  $\square$

The purpose of this note is to present four different combinatorial interpretations for the coefficients  $\gamma_j$  that are consequences of results in the work of the author and Wachs in [9] and of the author in [8]. We present now one of these combinatorial interpretations, whose proof will be given in Section 2.

A *planar leaf-labeled binary tree* with label set  $[n]$  is a rooted tree (a priori without labels) in which the set of children of every internal node is a totally ordered set with exactly two elements (the left and right children) and where each leaf has been assigned a unique element from the set  $[n]$ . By a subtree in a rooted tree  $T$  we mean the rooted tree induced by the descendants of any node  $x$  of  $T$ , including and

rooted at  $x$ . We say that a planar leaf-labeled binary tree with label set  $[n]$  is *normalized* if in each subtree, the leftmost leaf is the one with the smallest label. We denote the set of normalized binary trees with label set  $[n]$  by  $\mathbf{Nor}_n$ . All normalized trees with leaf labels in  $[3]$  are illustrated in Figure 2.

A *right descent* in a normalized tree is an internal node that is the right child of its parent. For  $T \in \mathbf{Nor}_n$  we define  $\text{rdes}(T) := |\{\text{right descents of } T\}|$ . A *double right descent* is a right descent whose parent is also a right descent. We denote by  $\mathbf{NDRD}_n$  the set of trees in  $\mathbf{Nor}_n$  with no double right descents.

**Theorem 1.3.** For  $n \geq 1$  and  $j \in \{0, 1, \dots, \lfloor \frac{n-1}{2} \rfloor\}$ ,

$$\gamma_j(T_n(t)) = |\{T \in \mathbf{NDRD}_n \mid \text{rdes}(T) = j\}|.$$

As it is illustrated in Figure 2, there are two trees in  $\mathbf{NDRD}_3$  (for  $n = 3$  it happens to be equal to  $\mathbf{Nor}_3$ ) with  $\text{rdes}(T) = 0$  and one with  $\text{rdes}(T) = 1$ , corresponding to  $\gamma_0 = 2$  and  $\gamma_1 = 1$  respectively.

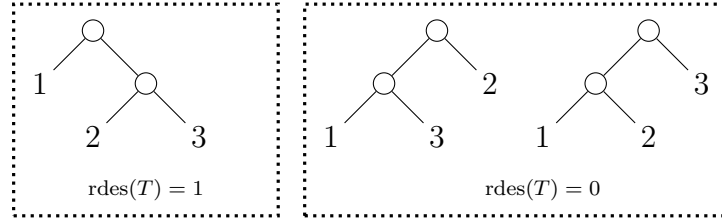


FIGURE 2. Set of normalized trees in  $\mathbf{NDRD}_3 = \mathbf{Nor}_3$

In Section 2 we provide the proof of Theorem 1.3 and an additional version of Theorem 1.3 also in terms of normalized trees but with a statistic different than  $\text{rdes}$ . In Section 3 we provide two additional versions of Theorem 1.3 in terms of the Stirling permutations introduced by Gessel and Stanley in [7]. In Section 4 we discuss a generalization of the  $\gamma$ -positivity of  $T_n(t)$  to the positivity of certain symmetric function in the basis of elementary symmetric functions.

## 2. COMBINATORIAL INTERPRETATIONS IN TERMS OF BINARY TREES

Now we consider normalized trees  $T$  where every internal node  $x$  of  $T$  has been assigned an element  $\mathbf{color}(x) \in \{0, 1\}$ . We call an element of this set of trees a *bicolored normalized tree* on  $[n]$ . A *bicolored comb* is a bicolored normalized tree  $T$  satisfying the following coloring restriction:

- (C) If  $x$  is a right descent of  $T$  then  $\mathbf{color}(x) = 0$  and  $\mathbf{color}(p(x)) = 1$ .

We denote by  $\mathbf{Comb}_n$  the set of bicolored combs and by  $\mathbf{Comb}_{n,i}$  the set of bicolored combs where  $i$  internal nodes have been colored 1 (and  $n - 1 - i$  colored 0). Figure 3 illustrates the bicolored combs on  $[3]$  grouped by the number of internal nodes that have been colored 1.

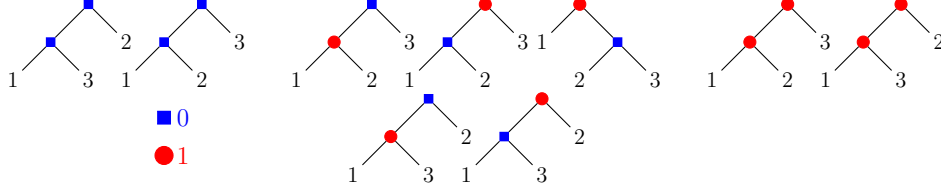


FIGURE 3. Set of bicolored combs on  $[3]$

Denote by  $\tilde{T} \in \mathbf{Nor}_n$  the underlying uncolored normalized tree associated to a tree  $T \in \mathbf{Comb}_n$ . Note that the coloring condition (C) implies that  $\tilde{T} \in \mathbf{NDRD}_n$ . Indeed, in a double right descent the coloring condition (C) cannot be satisfied since the parent of a double right descent is also a right descent. Also note that the monochromatic combs in  $\mathbf{Comb}_{n,0}$  and  $\mathbf{Comb}_{n,n-1}$  are just the traditional left combs that are described in [16] and that index a basis for the space  $\mathcal{Lie}(n)$ , the multilinear component of the free Lie algebra over  $\mathbb{C}$  on  $n$  generators (see [16] for details). Liu [10] and Dotsenko-Khoroshkin [4] independently proved a conjecture of Feigin regarding the dimension of the multilinear component  $\mathcal{Lie}_2(n)$  of the free Lie algebra with two compatible brackets, a generalization of  $\mathcal{Lie}(n)$ .

**Theorem 2.1** ([4, 10]). *For  $n \geq 1$ ,  $\dim \mathcal{Lie}_2(n) = |\mathcal{T}_n|$ .*

In particular, the space  $\mathcal{Lie}_2(n)$  has the decomposition

$$\mathcal{Lie}_2(n) = \bigoplus_{i=0}^{n-1} \mathcal{Lie}_2(n, i),$$

where the subspace  $\mathcal{Lie}_2(n, i)$  is the component generated by certain “bracketed permutations” with exactly  $i$  brackets of one of the types. Liu finds the following formula for the dimension of  $\mathcal{Lie}_2(n, i)$ .

**Theorem 2.2** ([10, Proposition 11.3]). *For  $n \geq 1$  and  $i \in \{0, 1, \dots, n-1\}$ ,*

$$\dim \mathcal{Lie}_2(n, i) = |\mathcal{T}_{n,i}|.$$

In [9] the author and Wachs studied the relation between  $\mathcal{Lie}_2(n, i)$  and the cohomology of the maximal intervals of a poset of weighted

partitions. Using poset topology techniques they found the following alternative description for the dimension of  $\mathcal{L}ie_2(n, i)$ .

**Theorem 2.3** ([9, Section 5]). *For  $n \geq 1$  and  $i \in \{0, 1, \dots, n-1\}$ ,*

$$\dim \mathcal{L}ie_2(n, i) = |\mathbf{Comb}_{n,i}|.$$

**Corollary 2.4.** *For every  $n \geq 1$  and  $i \in \{0, \dots, n-1\}$ ,*

$$|\mathbf{Comb}_{n,i}| = |\mathcal{T}_{n,i}|.$$

**Open Problem 2.5.** *Find an explicit bijection  $\mathbf{Comb}_{n,i} \rightarrow \mathcal{T}_{n,i}$ , for every  $n \geq 1$  and  $i \in \{0, \dots, n-1\}$ .*

**Theorem 2.6** (Theorem 1.3). *For  $n \geq 1$  and  $j \in \{0, 1, \dots, \lfloor \frac{n-1}{2} \rfloor\}$ ,*

$$\gamma_j(T_n(t)) = |\{T \in \mathbf{NDRD}_n \mid \text{rdes}(T) = j\}|.$$

*Proof.* First note that by the comments above if  $T \in \mathbf{Comb}_n$  then its underlying uncolored tree  $\tilde{T} \in \mathbf{NDRD}_n$ .

For a tree  $T \in \mathbf{Comb}_n$  call  $\text{free}(T)$  the number of internal nodes that are not right descents and whose right child is a leaf. Then in  $\mathbf{NDRD}_n$  we have that

$$\text{free}(T) + 2 \text{rdes}(T) = n - 1.$$

Over the set of bicolored combs with  $m$  free nodes there is a free action of  $(\mathbb{Z}_2)^m$  by toggling the colors of the free nodes. Then there are  $2^m$  bicolored combs with the same underlying tree  $\tilde{T} \in \mathbf{NDRD}_n$ . By Corollary 2.4 we can write  $T_n(t)$  as

$$\begin{aligned} T_n(t) &= \sum_{i=0}^{n-1} |\mathcal{T}_{n,i}| t^i \\ &= \sum_{i=0}^{n-1} |\mathbf{Comb}_{n,i}| t^i \\ &= \sum_{T \in \mathbf{Comb}_n} t^{|\{x \text{ internal in } T \mid \text{color}(x)=1\}|} \\ &= \sum_{\mathfrak{T} \in \mathbf{NDRD}_n} \sum_{\substack{T \in \mathbf{Comb}_n \\ \tilde{T}=\mathfrak{T}}} t^{|\{x \text{ internal in } T \mid \text{color}(x)=1\}|} \\ &= \sum_{\mathfrak{T} \in \mathbf{NDRD}_n} t^{\text{rdes}(\mathfrak{T})} (1+t)^{\text{free}(\mathfrak{T})} \\ &= \sum_{\mathfrak{T} \in \mathbf{NDRD}_n} t^{\text{rdes}(\mathfrak{T})} (1+t)^{n-1-2 \text{rdes}(\mathfrak{T})}. \end{aligned} \quad \square$$

**2.1. A second description in terms of normalized trees.** Define now the *valency*  $v(x)$  of a node (internal or leaf)  $x$  of  $T \in \mathbf{Nor}_n$  to be the minimal label in the subtree of  $T$  rooted at  $x$ . For an internal node  $x$  of  $T$  let  $L(x)$  and  $R(x)$  denote the left and right children of  $x$  respectively. A *Lyndon node* is an internal node  $x$  of  $T$  such that

$$(2.1) \quad v(R(L(x))) > v(R(x)).$$

A *Lyndon tree* is a normalized tree in which all its internal nodes are Lyndon. We denote  $\text{nlyn}(T)$  the number of non-Lyndon nodes in  $T$ . A *double non-Lyndon node* is a non-Lyndon node that is the left child of its parent and its parent is also a non-Lyndon node. We denote the set of trees in  $\mathbf{Nor}_n$  with no double non-Lyndon nodes by  $\mathbf{NDNL}_n$ . A *bicolored Lyndon tree* is a bicolored normalized tree satisfying the coloring condition:

- (L) For every non-Lyndon node  $x$  of  $T$  then  $\mathbf{color}(x) = 0$  and  $\mathbf{color}(L(x)) = 1$ .

The set of bicolored Lyndon trees is denoted  $\mathbf{Lyn}_n$  and the set of the ones with exactly  $i$  nodes with color 1 is denoted  $\mathbf{Lyn}_{n,i}$ .

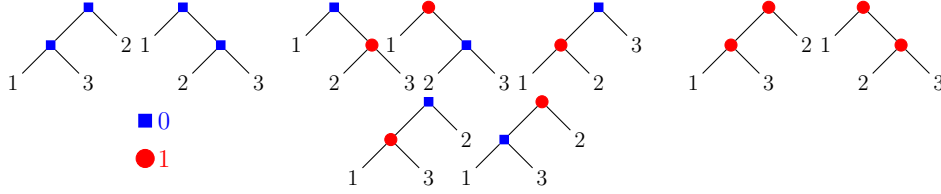


FIGURE 4. Set of bicolored Lyndon trees on  $[3]$

**Theorem 2.7** ([9, Section 5]). *For every  $n \geq 1$  and  $i \in \{0, \dots, n-1\}$ ,*

$$\dim \mathcal{Lie}_2(n, i) = |\mathbf{Lyn}_{n,i}|.$$

Hence,

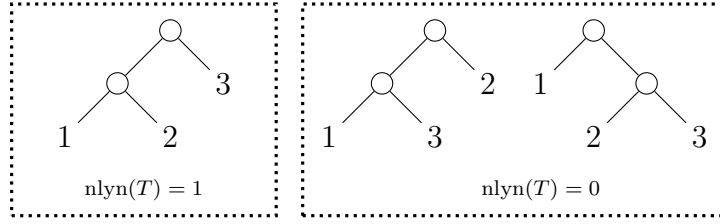
$$|\mathbf{Lyn}_{n,i}| = |\mathcal{T}_{n,i}|.$$

The proof of the following theorem follows the same arguments of the proof of Theorem 1.3.

**Theorem 2.8.** *For  $n \geq 1$  and  $j \in \{0, 1, \dots, \lfloor \frac{n-1}{2} \rfloor\}$ ,*

$$\gamma_j(T_n(t)) = |\{T \in \mathbf{NDNL}_n \mid \text{nlyn}(T) = j\}|.$$



FIGURE 5. Set of normalized trees in  $\text{NDNL}_3 = \text{Nor}_3$ 

### 3. COMBINATORIAL INTERPRETATION IN TERMS OF STIRLING PERMUTATIONS

Consider now the set of multipermutations of the multiset  $\{1, 1, 2, 2, \dots, n, n\}$  such that all numbers between the two occurrences of any number  $m$  are larger than  $m$ . To this family belongs for example the permutation 12234431 but not 11322344 since 2 is less than 3 and 2 is between the two occurrences of 3. This family (denoted  $\mathcal{Q}_n$ ) of permutations was introduced by Gessel and Stanley in [7] and the permutations in  $\mathcal{Q}_n$  are known as *Stirling Permutations*.

For a permutation  $\theta = \theta_1\theta_2\dots\theta_{2n}$  in  $\mathcal{Q}_n$  we say that the position  $i$  contains a *first occurrence* of a letter if  $\theta_j \neq \theta_i$  for all  $j < i$ , otherwise we say that it contains a *second occurrence*. An *ascending adjacent pair* in  $\theta$  is a pair  $(a, b)$  such that  $a < b$  and in  $\theta$  the second occurrence of  $a$  is the immediate predecessor of the first occurrence of  $b$ . An *ascending adjacent sequence (of length 2)* is a sequence  $a < b < c$  such that  $(a, b)$  and  $(b, c)$  are both ascending adjacent pairs. For example, in  $\theta = 13344155688776$  the ascending adjacent pairs are  $(1, 5)$ ,  $(5, 6)$  and  $(3, 4)$  but the only ascending adjacent sequence is  $1 < 5 < 6$ . We denote  $\text{NAAS}_n$  the set of all Stirling permutations in  $\mathcal{Q}_n$  that do not contain ascending adjacent sequences. Similarly, a *terminally nested pair* in  $\theta$  is a pair  $(a, b)$  such that  $a < b$  and in  $\theta$  the second occurrence of  $a$  is the immediate successor of the second occurrence of  $b$ . A *terminally nested sequence (of length 2)* is a sequence  $a < b < c$  such that  $(a, b)$  and  $(b, c)$  are both terminally nested pairs. For example, in  $\theta = 13443566518877$  the terminally nested pairs are  $(1, 5)$ ,  $(5, 6)$  and  $(3, 4)$  but the only terminally nested sequence is  $1 < 5 < 6$ . We denote  $\text{NTNS}_n$  the set of all Stirling permutations in  $\mathcal{Q}_n$  that do not contain terminally nested sequences. For  $\sigma \in \mathcal{Q}_n$ , we denote  $\text{aapair}(\sigma)$  the number of ascending adjacent pairs in  $\sigma$  and  $\text{tnpair}(\sigma)$  the number of terminally nested pairs in  $\sigma$ .

The following result in [8] relates the statistics above in  $\mathcal{Q}_{n-1}$  with the ones previously discussed for  $\text{Nor}_n$ .

**Proposition 3.1** ([8, Proposition 4.8]). *There is a bijection  $\phi : \mathbf{Nor}_n \rightarrow \mathcal{Q}_{n-1}$  such that for every  $T \in \mathbf{Nor}_n$ ,*

- (1)  $\text{rdes}(T) = \text{tnpair}(\phi(T))$
- (2)  $\text{nlyn}(T) = \text{aapair}(\phi(T))$
- (3)  $\phi(\mathbf{NDRD}_n) = \mathbf{NAAS}_{n-1}$
- (4)  $\phi(\mathbf{NDNL}_n) = \mathbf{NTNS}_{n-1}$ .

**Corollary 3.2.** *For  $n \geq 1$  and  $j \in \{0, 1, \dots, \lfloor \frac{n-1}{2} \rfloor\}$ ,*

$$\begin{aligned} \gamma_j(T_n(t)) &= |\{T \in \mathbf{NTNS}_{n-1} \mid \text{tnpair}(T) = j\}| \\ &= |\{T \in \mathbf{NAAS}_{n-1} \mid \text{aapair}(T) = j\}|. \end{aligned}$$

**Example 3.3.** The Stirling permutations in  $\mathcal{Q}_2$  are 1122, 1221 and 2211. In this particular case  $\mathcal{Q}_2 = \mathbf{NAAS}_2 = \mathbf{NTNS}_2$  and the statistics in Table 1 imply that  $\gamma_0 = 2$  and  $\gamma_1 = 1$  are the  $\gamma$ -coefficients of the polynomial  $\mathcal{T}_3(t)$ .

$\sigma$	tnpair	aapair
1122	0	1
1221	1	0
2211	0	0

TABLE 1. tnpair and aapair statistics in  $\mathcal{Q}_2$ .

#### 4. A COMMENT ABOUT $\gamma$ -POSITIVITY AND $e$ -POSITIVITY

Let  $\mathbf{x} = x_1, x_2, \dots$  be an infinite set of variables and  $\Lambda = \Lambda_{\mathbb{Q}}$  the ring of symmetric functions with rational coefficients on the variables  $\mathbf{x}$ , that is, the ring of power series on  $\mathbf{x}$  of bounded degree that are invariant under permutation of the variables.

Define  $e_0 := 1$ , for  $n \geq 1$

$$e_n := \sum_{1 \leq i_1 < i_2 < \dots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n},$$

and for an integer partition  $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$  (i.e., a weakly decreasing finite sequence of positive integers)  $e_\lambda := \prod_i e_{\lambda_i}$ . We call  $e_\lambda$  the *elementary symmetric function* corresponding to the partition  $\lambda$ .

It is known that for  $n \geq 0$ , the set  $\{e_\lambda \mid \lambda \vdash n\}$  is a basis for the  $n$ -th homogeneous graded component of  $\Lambda$ , where the grading is with respect to degree. See [11] and [14] for more information about symmetric functions.

Note that if we make the specialization  $x_i \mapsto 0$  in  $\Lambda$  for all  $i \geq 3$  then  $e_1 \mapsto x_1 + x_2$ ,  $e_2 \mapsto x_1 x_2$  and  $e_i \mapsto 0$  for all  $i \geq 3$ . Thus for a

partition  $\lambda$  of  $n$  (i.e.,  $\sum_i \lambda_i = n$ ) the symmetric function  $e_\lambda \mapsto 0$  unless  $\lambda = (2^j, 1^{n-2j})$  for some  $j \in \mathbb{N}$ . In that case,

$$e_{(2^j, 1^{n-2j})} \mapsto (x_1 x_2)^j (x_1 + x_2)^{n-2j}.$$

If we further replace  $x_1 \mapsto 1$  and  $x_2 \mapsto t$  we obtain

$$e_{(2^j, 1^{n-2j})} \mapsto t^j (1+t)^{n-2j}.$$

In other words, the elementary basis in two variables is equivalent to the  $\gamma$  basis. A consequence of this observation is that another possible approach to conclude the  $\gamma$ -positivity of a palindromic polynomial  $f(t)$  is to find an  $e$ -positive symmetric function  $F(x_1, x_2, \dots)$  such that  $f(t) = F(1, t, 0, 0, \dots)$ .

**4.1. Colored combs and comb type of a normalized tree.** A *colored comb* is a normalized binary tree  $T$  together with a function **color** that assigns positive integers in  $\mathbb{P}$  to the internal nodes of  $T$  and that satisfies the following coloring restriction: for each internal node  $x$  whose right child  $R(x)$  is not a leaf,

$$(4.1) \quad \mathbf{color}(x) > \mathbf{color}(R(x)).$$

Note that the set of colored combs that only use the colors 1 and 2 are the same as the bicolored combs defined in Section 2. We denote  $\mathbf{MComb}_n$  the set of colored combs with  $n$  leaves. Figure 6 shows an example of a colored comb.

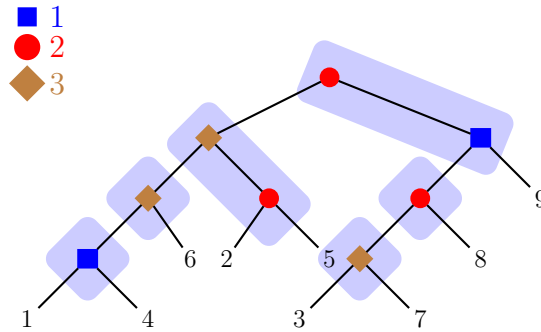


FIGURE 6. Example of a colored comb of comb type  $(2, 2, 1, 1, 1, 1)$

We can associate a type to each  $\Upsilon \in \mathbf{Nor}_n$  in the following way: Let  $\pi(\Upsilon)$  be the finest (set) partition of the set of internal nodes of  $\Upsilon$  satisfying

- for every pair of internal nodes  $x$  and  $y$  such that  $y$  is a right child of  $x$ ,  $x$  and  $y$  belong to the same block of  $\pi(\Upsilon)$ .

We define the *comb type*  $\lambda(\Upsilon)$  of  $\Upsilon$  to be the integer partition whose parts are the sizes of the blocks of  $\pi(\Upsilon)$ .

Note that the coloring condition (4.1) is closely related to the comb type of a normalized tree. The coloring condition implies that in a colored comb  $\Upsilon$  there are no repeated colors in each block  $B$  of the partition  $\pi(\Upsilon)$  associated to  $\Upsilon$ . So after choosing  $|B|$  different colors for the internal nodes of  $\Upsilon$  in  $B$ , there is a unique way to assign the colors such that  $\Upsilon$  is a colored comb (the colors must decrease towards the right in each block of  $\pi(\Upsilon)$ ). In Figure 6 this relation is illustrated.

For a colored comb  $C$  denote  $\mu(C)$  the sequence of nonnegative integers such that

$$\mu(C)(j) := |\{x \text{ a internal node in } C \mid \mathbf{color}(x) = j\}|.$$

Let  $\mathbf{x}^\mu := x_1^{\mu(1)} x_2^{\mu(2)} \dots$  and

$$F_{\mathbf{MComb}_n}(\mathbf{x}) := \sum_{C \in \mathbf{MComb}_n} \mathbf{x}^{\mu(C)}.$$

The following theorem is a consequence of the definition of a colored comb, the definition of the symmetric functions  $e_i(\mathbf{x})$  and the observations above (see [8]).

**Theorem 4.1** ([8]). *For  $n \geq 1$*

$$F_{\mathbf{MComb}_n}(\mathbf{x}) = \sum_{T \in \mathbf{Nor}_n} e_{\lambda(T)}(\mathbf{x}).$$

Note that  $F_{\mathbf{MComb}_n}(1, t, 0, 0, \dots) = \sum_{C \in \mathbf{Comb}_n} t^{\text{red } C} = T_n(t)$  and so Theorem 4.1 is a generalization of Theorem 1.3.

*Remark 4.2.* Versions of Theorem 4.1 can also be given in terms of a completely different type on the set  $\mathbf{Nor}_n$  corresponding to a family of multicolored Lyndon trees and also in terms of colored Stirling permutations, see [8].

## REFERENCES

- [1] P. Brändén. Sign-graded posets, unimodality of  $W$ -polynomials and the Charney-Davis conjecture. *Electron. J. Combin.*, 11(2):Research Paper 9, 15 pp. (electronic), 2004/06.
- [2] P. Brändén. Actions on permutations and unimodality of descent polynomials. *European J. Combin.*, 29(2):514–531, 2008.
- [3] P. Brändén. Unimodality, log-concavity, real-rootedness and beyond. *ArXiv preprint arXiv:1410.6601*, 2014.
- [4] V. V. Dotsenko and A. S. Khoroshkin. Character formulas for the operad of a pair of compatible brackets and for the bi-Hamiltonian operad. *Funktsional. Anal. i Prilozhen.*, 41(1):1–22, 96, 2007.

- [5] B. Drake. *An inversion theorem for labeled trees and some limits of areas under lattice paths*. ProQuest LLC, Ann Arbor, MI, 2008. Thesis (Ph.D.)—Brandeis University.
- [6] S. R. Gal. Real root conjecture fails for five- and higher-dimensional spheres. *Discrete Comput. Geom.*, 34(2):269–284, 2005.
- [7] I. Gessel and R. P. Stanley. Stirling polynomials. *J. Combinatorial Theory Ser. A*, 24(1):24–33, 1978.
- [8] R. S. González D’León. On the free Lie algebra with multiple brackets. *arXiv preprint arXiv:1408.5415*, 2014.
- [9] R. S. González D’León and M. L. Wachs. On the (co) homology of the poset of weighted partitions. *arXiv preprint arXiv:1309.5527*, 2013.
- [10] F. Liu. Combinatorial bases for multilinear parts of free algebras with two compatible brackets. *J. Algebra*, 323(1):132–166, 2010.
- [11] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, second edition, 1995. With contributions by A. Zelevinsky, Oxford Science Publications.
- [12] A. Postnikov, V. Reiner, and L. Williams. Faces of generalized permutohedra. *Doc. Math.*, 13:207–273, 2008.
- [13] L. W. Shapiro, W. J. Woan, and S. Getu. Runs, slides and moments. *SIAM J. Algebraic Discrete Methods*, 4(4):459–466, 1983.
- [14] R. P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.
- [15] R. P. Stanley. *Enumerative combinatorics. Volume 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2012.
- [16] M. L. Wachs. On the (co)homology of the partition lattice and the free Lie algebra. *Discrete Math.*, 193(1-3):287–319, 1998. Selected papers in honor of Adriano Garsia (Taormina, 1994).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KENTUCKY, LEXINGTON,  
KY 40506-0027

*E-mail address:* rafaeldeleon@uky.edu